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Problem of a phase transition in a one-dimensional Fermi system in the many-field limit

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Abstract. A one-dimensional model of a metal is generalized by assigning a component index to the electron creation and annihilation operators. In the limit in which the number of components n goes to infinity it is found that a mean-field-type phase transition occurs at a finite temperature even for short-ranged interactions. It is shown that this result is not in contradiction with general theorems concerning ODLRO.

1. Introduction

The model of a one-dimensional metal investigated by Bychkov *et al* (1966) and more recently by Menyhárd and Sólyom (1973), Sólyom (1973) and others (Fukuyama *et al* 1974, Luther and Emery 1974) cannot be solved exactly—except for a very special strong coupling case (Luther and Emery 1974)—and until now no completely satisfactory approximation scheme has been set up for its treatment. Therefore it is of some interest to construct generalized versions of the model which are solvable in a limiting case. Namely, we will assign a new component index to the electron creation and annihilation operators and investigate the limit in which the number of components n goes to infinity.

The resulting model exhibits a phase transition in the sense that ODLRO sets in at some finite temperature for attractive short-ranged interactions; the nature of the phase transition depends on the way the interaction potential is generalized. The possibility of a phase transition in one dimension with short ranged interactions is traced back to the fact that in the $n \to \infty$ limit of the theory all fluctuations vanish and the mean-field behaviour becomes exact. For any finite n, however, ODLRO should be lacking. The Ginzburg–Landau free energy functional of the problem is used to demonstrate that such a result cannot be obtained by a $1/n$ expansion starting from the $n \to \infty$ result. This is connected with the fact that k values lying in the range $0 < k < 1/n$ are responsible for destroying order. Bogolyubov's (1970) inequality, as used by Hohenberg (1967) in establishing the argument against two- and one-dimensional superconducting order, is shown not to be in contradiction with the above results.

2. The model and the $n \to \infty$ limit

For the details of the original model we refer to Menyhárd and Sólyom (1973). The

Hamiltonian of the n-component system is

$$H^{(n)} = H_0^{(n)} + H_{\text{int}}^{(n)} \qquad H_0^{(n)} = \sum_{k i \alpha} \epsilon_k a_{k\alpha i}^+ a_{k\alpha i}$$

$$H_{\text{int}}^{(n)} = \frac{1}{L} \sum_{ijkl} \sum_{k_1,k_2,k_3,k_4} \sum_{\alpha\beta} [g_{1ijkl}(k_1, \ldots, k_4) a_{k_1\alpha i}^+ a_{k_2 - 2k_F\beta j}^+ a_{k_3\beta k} a_{k_4 - 2k_F\alpha l} \tag{1}$$

$$+ g_{2ijkl}(k_1, \ldots, k_4) a_{k_1\alpha i}^+ a_{k_2 - 2k_F\beta j}^+ a_{k_3 - 2k_F\beta k} a_{k_4\alpha l}]$$

where $a_{k\alpha i}^+$ creates an electron of momentum k, spin component α and component i. k_F is the Fermi momentum and all the k_1, k_2, k_3, k_4 are near $+k_F$. L is the linear extension of the system and $\epsilon_k = k^2/2m - \mu$. The commutation rules are

$$\{a_{k\alpha i}^+, a_{k'\beta j}\} = \delta_{kk'}\delta_{\alpha\beta}\delta_{ij} \qquad \{a_{k\alpha i}, a_{k'\beta j}\} = 0.$$

No physical interpretation of the index i will be discussed here. For such a possibility see Efetov and Larkin (1974).

The interaction potentials $g_{1ijkl}(k_1, \ldots, k_4)$ ($g_{2ijkl}(k_1, \ldots, k_4)$) will be chosen to contain factors of two Kronecker deltas of the component indices and of constants $g_1(g_2)$ which correspond to interactions with momentum transfer $2k_F$ (0) in a narrow shell specified by ω_D around the Fermi 'surface' and zero outside. (A third type of coupling constant corresponding to the interaction of electrons on one side of the Fermi 'surface' with momentum transfer approximately zero could also have been included in our Hamiltonian. The $n \to \infty$ results are, however, unaffected by such an interaction.) g_1 and g_2 will be taken to be of order $1/n$. In this way three possible generalized interactions arise as shown in figure 1. It is easy to see that in cases (*a*), (*b*) and (*c*) closed loops of the

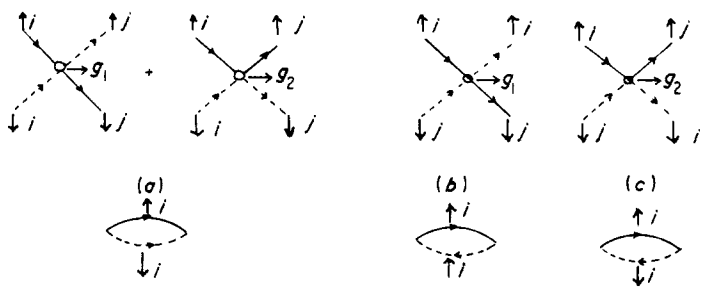

Figure 1. Generalized elementary vertices. Full line (broken line) indicates an electron propagating with momentum near $+k_F$ ($-k_F$). In process (*a*) the number of electrons of component index i is not conserved. In the limit of large n (*a*) enhances Cooper pair formation, (*b*) selects diagrams giving rise to a density wave of momentum $2k_F$ and (*c*) favours spin density wave-type diagrams. The corresponding characteristic 'logarithmic bubbles' are shown beneath.

Let us define the generalized susceptibility characteristic of Cooper pair formation by the analytic continuation to the upper ω half-plane of

$$\chi^{C}(k,\omega_\nu) = -\int_0^{1/T} d\tau \exp(i\omega_\nu\tau)\left\langle T_\tau\left[\frac{1}{\sqrt{n}}\sum_i\int\frac{dp}{2\pi}a_{p+k\uparrow i}(\tau)a_{-p\downarrow i}(\tau)\right.\right.$$
$$\left.\left.\times\frac{1}{\sqrt{n}}\sum_j\int\frac{dp'}{2\pi}a^{+}_{-p'\downarrow j}(0)a^{+}_{p'+k\uparrow j}(0)\right]\right\rangle \qquad k\simeq 0 \tag{2}$$

where $\omega_\nu = 2\pi\nu T$ (ν = integer). Let us take, for the sake of simplicity, $g_1 = g_2 \equiv g$ ($=\bar{g}/n$) in equation (1). The graphs contributing to χ^{C} are shown in figure 2. These can

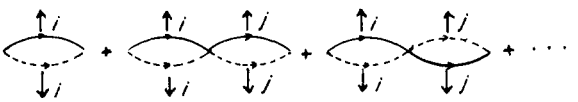

Figure 2. Diagrams contributing to the generalized pair susceptibility in leading order in n.

be summed easily in the usual way with the result to logarithmic accuracy

$$\chi^{C}(x) = -\frac{\ln x/\pi v}{1-(\bar{g}/\pi v)\ln x} \tag{3}$$

where $x = \max\{(k_B T/\omega_D),(\omega/\omega_D),(kv/\omega_D)\}$ and v is the Fermi velocity. The two other generalized susceptibilities, characterizing the density wave and spin density wave type responses of the system,

$$\chi^{DW}(k,\omega_\nu) = -\int_0^{1/T} d\tau \exp(i\omega_\nu\tau)\left\langle T_\tau\left[\frac{1}{\sqrt{n}}\sum_{i\alpha}\int\frac{dp}{2\pi}a^{+}_{p\alpha i}(\tau)a_{p+k\alpha i}(\tau)\right.\right.$$
$$\left.\left.\times\frac{1}{\sqrt{n}}\sum_{j\beta}\int\frac{dp'}{2\pi}a^{+}_{p'\beta j}(0)a_{p'-k\beta j}(0)\right]\right\rangle \qquad k\approx 2k_F$$
$$\chi^{SDW}(k,\omega_\nu) = -\int_0^{1/T} d\tau \exp(i\omega_\nu\tau)\left\langle T_\tau\left[\frac{1}{\sqrt{n}}\sum_i\int\frac{dp}{2\pi}a^{+}_{p\uparrow i}(\tau)a_{p+k\downarrow i}(\tau)\right.\right.$$
$$\left.\left.\times\frac{1}{\sqrt{n}}\sum_j\int\frac{dp'}{2\pi}a^{+}_{p'\downarrow j}(0)a_{p'-k\uparrow j}(0)\right]\right\rangle \qquad k\approx 2k_F$$

will not get enhanced by the interaction depicted in figure 1(*a*) in leading order in n.

For $k = \omega = 0$, the static pair susceptibility is obtained from equation (3) as

$$\chi^{C}\sim (T-T_c)^{-1}$$

with

$$k_B T_c = \omega_D \exp(-\pi v/|\bar{g}|). \tag{4}$$

This is exact in the $n\to\infty$ limit and shows a superconducting-type phase transition at T_c.

Below T_c the Green function equations of Gorkov (1958) are exact for the model in question in the $n\to\infty$ limit and the quantity

$$\Delta(r) = gk_B T\sum_i\sum_{\omega_n}F_{\omega_n i}(r,r) \qquad (\Delta = O(1)) \tag{5}$$

where F_i is the anomalous Green function defined in terms of the ith component of the particle operators and satisfies the same gap equation as the corresponding quantity does for $n = 1$ in the BCS approximation with T_c given by equation (4). As a result, the Ginzburg–Landau free energy functional of the problem

$$F[\Psi(r)] = \int \mathrm{d}r \left(a|\Psi|^2 + \tfrac{1}{2}b|\Psi|^4 + c\left|\frac{\partial \Psi}{\partial r}\right|^2 \right) \tag{6}$$

looks formally like the free energy functional of the $n = 1$ problem. The order parameter Ψ is proportional to Δ. The coefficients a, b and c are of order n. Moreover, equation (6), evaluated with the equilibrium value of Ψ, is exact for temperatures near the critical temperature in the $n \to \infty$ limit.

3. The effect of fluctuations for finite n

For finite n equation (6) may be considered as an approximate expression with the help of which fluctuation effects can be accounted for. In particular, the order parameter–order parameter correlation function G can be obtained for the present case generalizing the result of Rice (1965) in a straightforward manner ($R = |r_1 - r_2|$)

$$G(R) \sim \langle \Psi(r_1)\Psi(r_2) \rangle \sim [\Psi_0^2 + (\gamma q)^{-1} \exp(-qR)] \exp(-R/\gamma\Psi_0^2) \tag{7}$$

where $q = \xi^{-1}$ (ξ is the usual coherence length, $\xi^2 = c/2|a|$ and $\gamma = 4c/k_B T = \mathrm{O}(n)$. $\Psi_0^2 = -a/b$ being of O(1), the second term in the parentheses, the fluctuation in the absolute value of the order parameter, is of $\mathrm{O}(1/n)$ relative to the first one and thus vanishes in the limit $n \to \infty$ for all R. As to the factor coming from the phase fluctuations one has to be careful in what order the limits $n \to \infty$ and $R \to \infty$ are performed. If we take for fixed R the limit $n \to \infty$ (case A), we get

$$\exp(-R/\gamma\Psi_0^2) \to 1.$$

Thus in case A the previous result concerning the exactness of $F[\Psi_0]$ in the $n \to \infty$ limit is recovered. On the other hand, for any large but finite n we have in the $R \to \infty$ limit (case B)

$$\exp(-R/\gamma\Psi_0^2) \to 0.$$

It is tempting to conclude that, as all fluctuations vanish in the $n \to \infty$ limit, there is a phase transition in the sense of ODLRO and the phase transition is described exactly by mean-field theory. For any finite n, however, the phase fluctuations suppress the off-diagonal long-range order. Nevertheless, according to equation (7), a quasi-off-diagonal long-range order persists and extends to distances increasing with n as determined by the phase fluctuation coherence length $\gamma\Psi_0^2 = \mathrm{O}(n)$. As to a $1/n$ expansion around the $n \to \infty$ limit, effects connected with k values lying in the range $0 < k < 1/n$ cannot be accounted for by it and thus it can clearly not be used to find the correct answer as to ODLRO. This is also reflected by the behaviour of the order parameter renormalized by phase fluctuations

$$\langle \Psi \rangle = \Psi_0 \exp[-(2/\gamma\Psi_0^2 L) \sum_k 1/k^2] \tag{8}$$

in the above two types of limiting cases. If we take $k_{\min} \sim 1/n < 1/L$ in the summation, corresponding to case A, $\Sigma\, 1/k^2 \sim Ln$ is obtained and Ln cancels from the exponent, then $\langle \Psi \rangle$ is finite. On the other hand, taking the thermodynamic limit first (case B),

$\langle\Psi\rangle = 0$ is obtained. In order to arrive at a form similar to equation (8) all orders of $1/n$ have to be summed up; the first few terms are wholly misleading.

4. Discussion

As to the Bogolyubov (1970) inequality and Hohenberg's (1967) argument concerning the impossibility of a superconducting type order in one dimension, it has to be noted first that our generalization does not affect the validity of the f-sum rule provided the sum of the densities of momentum k, $\rho_k = \Sigma_i \rho_{ki}$ is used in the definition. (The total number of particles $N = \Sigma_i \Sigma_k \Sigma_\alpha a^+_{k\alpha i} a_{k\alpha i}$ commutes with the Hamiltonian.) The inequality exploited by Hohenberg (1967) is

$$C_{BB^+}(k) \geqslant 2T \frac{|\chi_{AB}(-k, k)|^2}{\chi_{A^+A}(-k)} \tag{9}$$

where C and χ denote correlation function and response function respectively. In the present case the choice

$$A = \mathrm{i}\frac{\partial}{\partial t}\rho_{-k}(t) \qquad B = \frac{1}{n}\sum_i B_i \qquad B_i = \sum_q S(q) a_{k-q\uparrow i} a_{q\downarrow i}$$

where $S(q)$ is a smearing function leads to $\chi_{A^+A} \sim nk^2$, $\chi_{AB} \sim \Delta$; C_{BB^+}, however, cannot be related either to the density correlation function or to any other physical quantity for which the integral over k remains finite for large n and thus equation (9) is useless. On the other hand, if $B = B_i$ is chosen, $\chi_{AB} \sim \Sigma_q S(q) a_{k-q\uparrow i} a_{q\downarrow i}$, which is proportional to Δ_i, the ith summand in equation (5), and $C_{BB^+}(k)$ can be related to the density correlation function of the ith component. Equation (9) can then be analysed analogously to the original $n = 1$ case with the only difference being in a factor of $1/n$, (coming from χ_{A^+A}), which appears on its right-hand side. Performing the limit $n \to \infty$ before the thermodynamic limit (case A) leads to the vanishing of the right hand side of equation (9) and thus the Δ_i can be finite in this limiting case.

It is interesting to compare the behaviour of the n-component fermion field (component index attached to the fermion operators) to the n-component classical (Ginzburg–Landau) field (component index attached to the order parameter) in the $n \to \infty$ limit (Imry and Scalapino 1974). The common feature is that a quasi-phase transition becomes a true phase transition in this limit and this takes place in a nonanalytic way in n^{-1}. There is, however, a significant difference, namely that the phase transition in the $n \to \infty$ limit is of first order for the classical field (Imry and Scalapino 1974), while it is of second order in the present case.

Finally let us note that the above considerations apply also for the interactions (b) and (c) of figure 1 with the only difference being in the nature of the phase transition for $n \to \infty$ and in the consequent difference in the definition of the order parameter.

Treatment of the problem in the framework of the renormalization group method will be given elsewhere.

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